EQUATIONS OF BIELLIPTIC MODULAR CURVES

JOSEP GONZÁLEZ

ABSTRACT. We give a procedure to determine equations for the modular curves $X_0(N)$ which are bielliptic and equations for the 30 values of N such that $X_0(N)$ is bielliptic and nonhyperelliptic are presented.

1. Introduction

A curve X (smooth and projective) of genus $g > 1$ defined over a number field K is said to be hyperelliptic, resp. bielliptic, over K, if there is an involution w defined over K such that the quotient curve $Y = X/\langle w\rangle$ has genus zero, resp. genus one, and $Y(K) \neq \emptyset$. The last condition amounts to saying that Y admits a hyperelliptic model over K or the genus one quotient is an elliptic curve over K.

According to Abramovich and Harris (cf. [1]), we know that for a curve X/K of genus > 1 the set of the points $P \in X(\overline{K})$ lying in a quadratic extension of K contains infinitely many points if and only if X is hyperelliptic over K or X is bielliptic over K and the corresponding elliptic quotient curve has rank ≥ 1.

When we restrict to the modular curves $X_0(N)/\mathbb{Q}$, the cusp $\infty \in X_0(N)(\mathbb{Q})$ and, thus, $X_0(N)$ is hyperelliptic or bielliptic over $\mathbb{Q}$ if there exists an involution w defined over $\mathbb{Q}$ whose quotient curve has genus ≤ 1. In [9], Ogg determined the 19 values of N for which $X_0(N)$ is hyperelliptic over $\mathbb{Q}$. Later in [3], Bars determined the 41 values of N for which $X_0(N)$ is bielliptic over $\mathbb{Q}$. Next, we display the unique 30 values of N of all them for which $X_0(N)$ is non-hyperelliptic:

$$\begin{array}{c} 34, 38, 42, 43, 44, 45, 51, 53, 54, 55, 56, 60,\ \ 61,\ \ 62,\ \ 63, \\ 64, 65, 69, 72, 75, 79, 81, 83, 89, 92, 94, 95, 102, 119, 131. \end{array} \tag{1}$$

In [6], equations for the 19 hyperelliptic modular curves are presented. The goal of this article is to determine equations for these 30 bielliptic modular curves. In this way, for each modular curve $X_0(N)$ of genus > 1 we could determine almost all points lying in a quadratic field.

2. General facts on the modular curve $X_0(N)$

From now on, $X_0(N)$ denotes the algebraic curve over $\mathbb{Q}$ attached to the modular group $\Gamma_0(N)$ and such that $\mathbb{Q}(X_0(N))$ is the subfield of $\mathbb{C}(X_0(N))$ which consists of the functions with rational q-expansion at the cusp ∞, where $q = e^{2\pi i z}$. Next, we summarize some well-known facts which shall be used in the next section.

2.1. **The group** $\mathrm{Aut}(X_0(N))$**.** The group $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ is the group of the automorphisms of the complex upper half-plane. Let us denote by $\Gamma_0^*(N)$ the normalizer of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. The group $B(N) = \Gamma_0^*(N)/\Gamma_0(N)$ provides a subgroup of $\mathrm{Aut}(X_0(N))$ described by Lehner and Newman in [8] and, later, revised by Atkin and Lehner in [2].

Let e_2 and e_3 be the greatest exponents such that $2^{2e2}\cdot 3^{e_3}$ divides N. Set $\nu_2 = 2^{\min(3,\lfloor e_2/2\rfloor)}$ and $\nu_3 = 3^{\min(1,\lfloor e_3/2\rfloor)}$. For every positive divisor d of ν_2 or ν_3, the matrix $\begin{pmatrix} 1 & 1/d \\ 0 & 1 \end{pmatrix} \in \Gamma_0^*(N)$ and, thus, provides an automorphism of $X_0(N)$ which will denoted by S_d.

For any positive divisor d of N coprime to N/d, the matrix $\frac{1}{\sqrt{d}}\begin{pmatrix} A\cdot d & B \\ N\cdot C & D\cdot d \end{pmatrix}$ with determinant 1 and $A, B, C, D \in \mathbb{Z}$ lies in $\Gamma_0(N)^*$ and provides an involution w_d on $X_0(N)$ independent on A, B, C and D, called the Atkin-Lehner involution attached to d. We denote by $W(N)$ the set of the Atkin-Lehner

The author is partially supported by DGICYT Grant MTM2009-13060-C02-02.

2010 *Mathematics Subject Classification*: 11F03, 14H45.

involutions, which is a commutative group since $w_{d_1} \cdot w_{d_2} = w_{d_1 \cdot d_2/\gcd(d_1,d_2)^2}$. The group $B(N)$ is generated by the group $W(N)$, S_{ν_2} and S_{ν_3}. In [7], Kenku and Momose proved that when the genus of $X_0(N)$ is >1 and $N \neq 37, 63$ one has that $\mathrm{Aut}(X_0(N)) = B(N)$.

Lemma 2.1. *The field of definition of any Atkin-Lehner involution and S_2 is $\mathbb{Q}$ and for $d > 2$ the field of definition of S_d is $\mathbb{Q}(\zeta_d)$, where ζ_d is a primitive d-th root of unity.*

Proof. For an Atkin-Lehner involution w_d, one has that $w_d^*(\mathbb{Q}(X_0(N)) = \mathbb{Q}(X_0(N))$ and, thus, w_d is defined over $\mathbb{Q}$. Indeed, the function field $\mathbb{Q}(X_0(N))$ is generated by the functions $j(z)$, $j(N\,z)$ and w_d sends these functions to the functions $j(d\,z)$, $j(N/d\,z)$, which lie in $\mathbb{Q}(X_0(N))$. It is immediate to check that for S_d the number field $\mathbb{Q}(\zeta_d)$ is the smallest number field K such that $K \otimes \mathbb{Q}(X_0(N))$ contains $S_d^*(\mathbb{Q}(X_0(N)))$. □

2.2. **Cusp forms of weight two.** We recall that we can identify the $\mathbb{C}$-vector space of weight 2 cusp forms on $\Gamma_0(N)$, i.e. $S_2(\Gamma_0(N))$, with $\Omega^1_{X_0(N)/\mathbb{C}}$ via the map $f(q) \mapsto f(q)\,\frac{dq}{q}$. Moreover, via this map $\Omega^1_{X_0(N)/\mathbb{Q}}$ is in bijective correspondence to the set of weight two cusp forms with rational q-expansion.

Let M be a positive divisor of N. For any positive divisor d of N/M, the map on the complex upper half-plane given by $z \mapsto dz$ provides a nonconstant morphism $B_d \colon X_0(N) \to X_0(M)$ which acts on the cusp forms of weight two by sending $f(q) \in S_2(\Gamma_0(M))$ to $f(q^d) \in S_2(\Gamma_0(N))$. The vector space $S_2(\Gamma_0(N))^{\mathrm{old}}$ is defined as the sum of the images of such maps for all $M|N$ and $d|N/M$. The vector space $S_2(\Gamma_0(N))$ has a hermitian inner product called the Petersson inner product and the vector space $S_2(\Gamma_0(N))^{\mathrm{new}}$ is defined as the orthogonal complement to $S_2(\Gamma_0(N))^{\mathrm{old}}$. We denote by New_N the set of normalized cusp forms in $S_2(\Gamma_0(N))^{\mathrm{new}}$ which are eigenvectors of all Hecke operators and Atkin-Lehner involutions. By a normalized cusp form we mean a cusp form whose first non-zero Fourier coefficient is equal to 1. It is well-known that New_N is a basis of $S_2(\Gamma_0(N))^{\mathrm{new}}$.

In Table 5 of [5], it can be found the dimensions of the vector spaces $S_2(\Gamma_0(N))^G$ and $(S_2(\Gamma_0(N))^{\mathrm{new}})^G$ for any subgroup G of the group of the Atkin-Lehner involutions $W(N)$ for $N \leq 300$. The following result will be useful in order to determine weight two cusp forms invariants under an Atkin-Lehner involution.

Lemma 2.2. *Let M and N be positive integers such that $M|N$. Let M_1 be a positive divisor of M such that and* $\gcd(M, M/M_1) = 1$ *and let ℓ be a positive divisor of N/M such that* $\gcd(M_1\,\ell, N/(M_1\,\ell) = 1$. *If $f \in S_2(\Gamma_0(M))$ is a normalized eigenvector of the Atkin-Lehner involution w_{M_1} with eigenvalue $\varepsilon(f)$ and $\varepsilon \in \{-1, 1\}$, then $f(q) + \varepsilon\, f(q^\ell) \in S_2(\Gamma_0(N))$ is a normalized eigenvector of the Atkin-Lehner involution $w_{M_1\,\ell}$ with eigenvalue $\varepsilon(f) \cdot \varepsilon$.*

Proof. An automorphism u on $X_0(N)$ whose action on the upper half-plane is given by a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, sends a weight two cusp form h to $h(\gamma(z))\frac{1}{(Cz+D)^2}$. The statement follows from the fact that $w_{M_1\,\ell}$ sends f to $\varepsilon(f) f(q^\ell)$. □

2.3. **Modular parametrizations of elliptic curves.** Since Shimura-Taniyama-Weil's conjecture was proved, we know that for an elliptic curve $E/\mathbb{Q}$ there exist a positive integer N and a nonconstant morphism $\pi \colon X_0(N) \to E$ defined over $\mathbb{Q}$. Such a morphism π will be called a modular parametrization of E and level N. The following conditions are equivalent:

(i) The conductor of E is N.
(ii) There exists a modular parametrization π of E and level N such that $\pi^*(\Omega^1_E) \subset S_2(\Gamma_0(N))^{\mathrm{new}}\frac{dq}{q}$.
(iii) There exist $f \in \mathrm{New}_N$ and a modular parametrization π of E and level N such that $\pi^*(\Omega^1_{E/\mathbb{Q}}) = \mathbb{Q} \cdot f(q)\frac{dq}{q}$.

Such a parametrization will be called **new** of level N and the newform f in part (iii) is unique and determines the $\mathbb{Q}$-isogeny class of E.

For a non-new modular parametrization π of E and level N, the conductor M of E divides N and $\pi^*(\Omega^1_{E/\mathbb{Q}})$ is an one dimensional $\mathbb{Q}$-vector subspace of

$$H_{E,N} = \bigoplus_{d|N/M} \mathbb{Q} f(q^d)\frac{dq}{q} \subset S_2(\Gamma_0(N))^{\mathrm{old}}\,,$$

where f is the normalized newform of level M attached to E. In fact, for any nonzero cusp form $h \in H_{E,N}$ there exists a modular parametrization π of E and level N such that $\pi^*(\Omega^1_{E/\mathbb{Q}}) = \mathbb{Q} \cdot h(q)\,\frac{dq}{q}$.

A modular parametrization π of E and level N is called **optimal** if the morphism induced on their jacobians $\pi_*\colon\ \mathrm{Jac}(X_0(N)) \to E$ has connected kernel. If $\pi^*(\Omega^1_{E/\mathbb{Q}}) = \mathbb{Q}\cdot h(q)\,\frac{dq}{q}$ for some $h \in S_2(\Gamma_0(N))$, then the condition to be π optimal amounts to saying that the elliptic curve attached to the lattice

$$\Lambda = \left\{ \int_\gamma h(q)\frac{dq}{q} : \gamma \in H_0(X_0(N), \mathbb{Z}) \right\} \tag{2}$$

is $\mathbb{Q}$-isomorphic to E, i.e. $c_4(\Lambda) = \alpha^4\, c_4(E)$ and $c_6(\Lambda) = \alpha^6\, c_6(E)$ for some nonzero $\alpha \in \mathbb{Q}$. For another modular parametrization π_1 of an elliptic curve $E_1/\mathbb{Q}$ and level N such that $\pi^*(\Omega^1_{E/\mathbb{Q}}) = \pi_1^*(\Omega^1_{E_1/\mathbb{Q}})$, if π is optimal then there is an isogeny $\mu\colon E \to E_1$ defined over $\mathbb{Q}$ such that $\pi_1 = \mu \circ \pi$ and, in particular, $\deg \pi | \deg \pi_1$.

We will denote the $\mathbb{Q}$-isomorphism class of an elliptic curve $E/\mathbb{Q}$ by giving its conductor N and Cremona's label, i.e. a letter X and a positive integer. For instance, the elliptic curve $15A8$ stands for the elliptic curve of conductor 15 with Cremona's label $A8$. The conductor N and the letter X, for instance $15A$, denotes the $\mathbb{Q}$-isogeny class of E and f_{NX} will denote the attached newform to E. The optimal quotient in the $\mathbb{Q}$-isogeny class of E, called the strong Weil curve, is always labeled with the number 1.

We point out that Manin's conjecture has been checked for all strong Weil elliptic curves, i.e. optimal new modular parametrizations, in Cremona's tables. That is, if $y^2 + a_1 x\, y + a_3\, y = x^3 + a_2\, x^2 + a_4\, x + a_6$ is a minimal model over $\mathbb{Z}$ of a strong Weil elliptic curve E of conductor N, then there exists a new nodular parametrization π of level N for E such that

$$\pi^*\left(\frac{dx}{2\,y + a_1\,x + a_3}\right) = \pm f(q)\frac{dq}{q}\,,$$

where $f \in \mathrm{New}_N$ is the corresponding normalized newform attached to the $\mathbb{Q}$-isogeny class of E. In other words, if $\pi(\infty)$ is taken to be the infinity point of E, the q-expansions of the modular functions x, y are of the form:

$$x = \frac{1}{q^2} + \sum_{n \geq -1} a_n q^n\,, \quad y = \mp\frac{1}{q^3} + \sum_{n \geq -2} b_n q^n \quad \text{and} \quad a_n, b_n \in \mathbb{Q}\,.$$

Equivalently, when we replace h with f in (2), the lattice Λ obtained and the the minimal model of E have the same invariants c_4 and c_6.

3. Procedure to determine equations

From now on, N is a value in the list (1). Let w be a bielliptic involution defined over $\mathbb{Q}$, let $\pi\colon X_0(N) \to X_0(N)/\langle w\rangle$ be the natural projection and let us denote by E the elliptic curve $(X_0(N)/\langle w\rangle, \pi(\infty))$. Since $\deg \pi = 2$, the parametrization π is optimal. Now we split in three steps the procedure to find an equation for $X_0(N)$:

Step 1: Determination of E, w and the normalized cusp form h such that $\pi^*(\Omega^1_E) = \langle h(q)dq/q\rangle$. Since $X_0(N)$ can have several bielliptic involutions defined over $\mathbb{Q}$, first we will determine for which values of N we can take w such that π is new. Clearly, $X_0(N)$ is bielliptic with a new modular parametrization for an elliptic quotient E if and only if there exists an elliptic curve of conductor N with modular degree equal to 2.

By checking in Table 22 of [4] among the elliptic curves E with conductor N as in (1), we obtain that this fact occurs exactly for 22 values of N. For each of these values of N, we fix this elliptic curve E as the corresponding bielliptic quotient and we determine the involution $w \in \mathrm{Aut}_{\mathbb{Q}}(X_0(N))$ such that $(\mathrm{New}_N)^{\langle w\rangle} = h$, where h is the newform attached to E (see Table 1).

For the remaining 8 values of N, i.e. $N \in \{42, 60, 63, 72, 75, 81, 95, 119\}$, we fix a bielliptic involution w defined over $\mathbb{Q}$ among the given in Theorem 3.15 of [3]. More precisely, for $N \neq 72$ we choose w to be an Atkin-Lehner involution and for $N = 72$ we take $w = S_2$ (see Table 2). In order to find a normalized cusp form h such that $S_2(\Gamma_0(N))^{\langle w\rangle} = \langle h\rangle$, we proceed as follows. For $N \neq 72$, by applying Lemma 2.2, we can easily determine a newform $f \in \mathrm{New}_M$ with $M|N$ and and integer q-expansion such that

$S_2(\Gamma_0(N))^{\langle w\rangle} = (\bigoplus_{d|N/M} \mathbb{Q} f(q^d)\frac{dq}{q})^{\langle w\rangle} = \mathbb{Q}\, h$. The newform f only determines the $\mathbb{Q}$-isogeny class of E. In order to determine its $\mathbb{Q}$-isomorphism class, we compute the corresponding lattice Λ attached to h. This fact allow us to identify E in Cremona's tables (cf. Table 2). In all cases, Λ turns out to be the lattice corresponding to a minimal model of the elliptic curve E. For $N = 72$, the normalized cusp form h is $f(q^2)$, where f is the normalized newform of level 36 attached to the isogeny class $36A$ and E is the elliptic curve $36A1$.

Step 2: Determination of the q-expansions of the functions $x, y \in \mathbb{Q}(E)$. Let $F_N(x, y) = y^2 + a_1\, x\, y + a_3\, y - (x^3 + a_2\, x^2 + a_4\, x + a_6) \in \mathbb{Z}[x, y]$ be the polynomial such that $F(x, y) = 0$ is the minimal model given in Cremona's Tables for E. For each $N \neq 72$, the lattice obtained from the normalized cusp form h in $\pi^*(\Omega^1_{E/\mathbb{Q}}) \subset S_2(\Gamma_0(N))$ corresponds to the minimal model $F_N(x, y) = 0$. Therefore, we can take y such that the coefficients of the Fourier expansion of x and y are the form

$$x = \frac{1}{q^2} + \sum_{n\geq -1} a_n q^n\,, \quad y = \frac{1}{q^3} + \sum_{n\geq -2} b_n q^n\,.$$

The Fourier coefficients of these modular functions can be determined recursively by means of the equalities

$$y = -\left(\frac{q dh/dq}{h} + a_1\, x + a_3\right)/2 \qquad \text{and} \qquad F(x, y) = 0\,.$$

For $N = 72$, by proceeding similarly for the elliptic curve $36A1$ with respect its attached normalized newform $f \in \mathrm{New}_{36}$, we obtain the q-expansions of x and y as functions on $X_0(36)$. It is clear that the functions for our case are $x(q^2)$ and $y(q^2)$.

Step 3: Determination of a suitable generator of the extension $\mathbb{Q}(X_0(N))/\mathbb{Q}(E)$. Let $\mathcal{G}_N$ be the multiplicative group of the modular functions on $X_0(N)$ which are equal to $\prod_{1\leq d|N} \eta(dz)^{r_d}$ for some integers r_d and where $\eta(z)$ is the Dedekind function. The group $\mathcal{G}_N$ is the multiplicative subgroup of $\mathbb{Q}(X_0(N))$ which consists of the normalized functions whose zeros and poles are cusps (for a detailed description of this group see 2.2 of [6]). In our case, due to the fact that w left stable the set of cusps and $w \in \mathrm{Aut}_{\mathbb{Q}}(X_0(N))$, the involution w induces an involution w^* on $\mathbb{Q} \otimes \mathcal{G}_N$. By Proposition 2 of [6], there exists a function $u \in \mathcal{G}_N$ satisfying:

(i) The polar part of u is a multiple of the divisor (∞) or $(\infty) + (w(\infty))$.
(ii) div $w^*(u) \neq$ div u.

Once such a function u is chosen, the divisor of $w^*(u)$ is determined and we can find a function $v \in \mathcal{G}_N$ having the same divisor as $w^*(u)$, i.e. $w^*(u) = a\, v$ for some nonzero rational number a. By using the q-expansions of u, v, x and y, we can determine a because $u + a\, v$ must be equal to a polynomial with rational coefficients in the functions x and y. In fact, if w is an Atkin-Lehner involution ($N \neq 64, 72$), then Proposition 3 of [6] allows us to determine a without using q-expansions. In any case, with our choice, the rational number a turns out to be always integer. Finally, we take then function $t := u - a\, v$ which satisfies $w^*(t) = -t$. Therefore, $\mathbb{Q}(X_0(N)) = \mathbb{Q}(E)(t)$ and, moreover, the function t^2 lies in $\mathbb{Q}(E)$ and has a unique pole in $\pi(\infty)$. Hence,

$$t^2 = P(x, y)$$

for some polynomial P with integers coefficients, which provides an equation for $X_0(N)$ related with the chosen equation for E, for which its Mordell-Weil group is described in Cremona's tables. The polynomial $P(x, y)$ is taken as a polynomial of the form $P_1(x) + P_2(x)\, y$, where $P_1(x), P_2(x) \in \mathbb{Z}$. Since the degree of $P_1(x)$ agrees with $-\operatorname{ord}_\infty u$ and $\deg P_2(x) \leq \deg P_1(x) - 2$, u is chosen to be $-\operatorname{ord}_\infty u$ minimal. In tables 3 and 4 of the appendix, the corresponding functions t are exhibited for the new and non-new case respectively, while in Tables 5 and 6 the polynomials $P(x, y)$ are presented for the new and non-new case respectively.

4. Appendix

4.1. Tables for E, w and h.

Table 1 (new case)

N	w	$X_0(N)/\langle w\rangle$		$h(q)$
34	w_{17}	$34A1$	$y^2+xy=x^3-3x+1$	$f_{34A}(q)$
38	w_{19}	$38B1$	$y^2+yx+y=x^3+x^2+1$	$f_{38B}(q)$
43	w_{43}	$43A1$	$y^2+y=x^3+x^2$	$f_{43A}(q)$
44	w_{11}	$44A1$	$y^2=x^3+x^3+3x-1$	$f_{44A}(q)$
45	w_5	$45A1$	$y^2+yx=x^3-x^2-5$	$f_{45A}(q)$
51	w_{51}	$51A1$	$y^2+y=x^3+x^2+x-1$	$f_{51A}(q)$
53	w_{53}	$53A1$	$y^2+xy+y=x^3-x^2$	$f_{53A}(q)$
54	w_{27}	$54B1$	$y^2+xy+y=x^3-x^2+x-1$	$f_{54B}(q)$
55	w_{11}	$55A1$	$y^2+xy=x^3-x^2-4x+3$	$f_{55A}(q)$
56	w_7	$56A1$	$y^2=x^3+x+2$	$f_{56A}(q)$
61	w_{61}	$61A1$	$y^2+xy=x^3-2x+1$	$f_{61A}(q)$
62	w_{31}	$62A1$	$y^2+xy+y=x^3-x^2-x+1$	$f_{62A}(q)$
64	$(S_2\cdot w_{64})^2$	$64A1$	$y^2=x^3-4$	$f_{64A}(q)$
65	w_{65}	$65A1$	$y^2+xy=x^3-x$	$f_{65A}(q)$
69	w_{23}	$69A1$	$y^2+xy+y=x^3-x-1$	$f_{69A}(q)$
79	w_{79}	$79A1$	$y^2+xy+y=x^3+x^2-2x$	$f_{79A}(q)$
83	w_{83}	$83A1$	$y^2+xy+y=x^3+x^2+x$	$f_{83A}(q)$
89	w_{89}	$89A1$	$y^2+xy+y=x^3-x$	$f_{89A}(q)$
92	w_{23}	$92A1$	$y^2=x^3+x^2+2x+1$	$f_{92A}(q)$
94	w_{47}	$94A1$	$y^2+xy+y=x^3-x^2-1$	$f_{94A}(q)$
101	w_{101}	$101A1$	$y^2+y=x^3x^2-x-1$	$f_{101A}(q)$
131	w_{131}	$131A1$	$y^2+y=x^3-x^2+x$	$f_{131A}(q)$

Table 2 (non-new case)

N	w		$X_0(N)/\langle w\rangle$	$h(q)$
42	w_{14}	$21A4$	$y^2+xy=x^3+x$	$f_{21A}(q)+2f_{21A}(q^2)$
60	w_{15}	$20A2$	$y^2=x^3+x^2-x$	$f_{20A}(q)+3f_{20A}(q^3)$
63	w_{63}	$21A4$	$y^2+xy=x^3+x$	$f_{21A}(q)-3f_{21A}(q^3)$
72	S_2	$36A1$	$y^2=x^3+1$	$f_{36A}(q^2)$
75	w_{75}	$15A8$	$y^2+x\,y+y=x^3+x^2$	$f_{15A}(q)-5f_{15A}(q^5)$
81	w_{81}	$27A3$	$y^2+y=x^3$	$f_{27A}(q)-3f_{27A}(q^3)$
95	w_{95}	$19A3$	$y^2+y=x^3+x^2+x$	$f_{19A}(q)-5f_{19A}(q^5)$
119	w_{119}	$17A4$	$y^2+xy+y=x^3-x^2-x$	$f_{17A}(q)-7f_{17A}(q^7)$

4.2. **Tables for t.**

Table 3 (new case)

N	t
34	$\frac{\eta(2z)^4\eta(17z)^2}{\eta(z)^2\eta(34z)^4} - 17\frac{\eta(z)^2\eta(34z)^4}{\eta(2z)^4\eta(17z)^2}$
38	$\frac{\eta(2z)^8\eta(19z)^4}{\eta(z)^4\eta(38z)^8} - 19^2\frac{\eta(z)^4\eta(38z)^8}{\eta(2z)^8\eta(19z)^4}$
43	$\frac{\eta(z)^4}{\eta(43z)^4} - 43^2\frac{\eta(43z)^4}{\eta(z)^4}$
44	$\frac{\eta(4z)^4\eta(22z)^2}{\eta(2z)^2\eta(44z)^4} + 11\frac{\eta(2z)^2\eta(44z)^4}{\eta(4z)^4\eta(22z)^2}$
45	$\frac{\eta(9z)^3\eta(15z)}{\eta(3z)\eta(45z)^3} + 5\frac{\eta(3z)\eta(45z)^3}{\eta(9z)^3\eta(15z)}$
51	$\frac{\eta(3z)^9\eta(17z)^3}{\eta(z)^3\eta(51z)^9} + 17^3\frac{\eta(z)^3\eta(51z)^9}{\eta(3z)^9\eta(17z)^3}$
53	$\frac{\eta(z)^6}{\eta(53z)^6} - 53^3\frac{\eta(53z)^6}{\eta(z)^6}$
54	$\frac{\eta(18z)\eta(27z)^3}{\eta(9z)\eta(54z)^3} + \frac{\eta(z)^3\eta(6z)}{\eta(2z)^3\eta(3z)}$
55	$\frac{\eta(5z)^5\eta(11z)}{\eta(z)\eta(55z)^5} - 11^2\frac{\eta(z)\eta(55z)^5}{\eta(5z)^5\eta(11z)}$
56	$\frac{\eta(8z)^4\eta(28z)^2}{\eta(4z)^2\eta(56z)^4} + 7\frac{\eta(4z)^2\eta(56z)^4}{\eta(8z)^4\eta(28z)^2}$
61	$\frac{\eta(z)^2}{\eta(61z)^2} - 61\frac{\eta(61z)^2}{\eta(z)^2}$
62	$\frac{\eta(2z)^8\eta(31z)^4}{\eta(z)^4\eta(62z)^8} - 31^2\frac{\eta(z)^4\eta(62z)^8}{\eta(2z)^8\eta(31z)^4}$
64	$\frac{\eta(32z)^6}{\eta(16z)^2\eta(64z)^4} - 4\frac{\eta(16z)^2\eta(64z)^4}{\eta(32z)^6}$
65	$\frac{\eta(5z)^5\eta(13z)}{\eta(z)\eta(65z)^5} - 13^2\frac{\eta(5z)\eta(13z)^5}{\eta(z)^5\eta(65z)}$
69	$\frac{\eta(3z)^9\eta(23z)^3}{\eta(z)^3\eta(69z)^9} + 23^3\frac{\eta(z)^3\eta(69z)^9}{\eta(3z)^9\eta(23z)^3}$
79	$\frac{\eta(z)^4}{\eta(79z)^4} - 79^2\frac{\eta(79z)^4}{\eta(z)^4}$
83	$\frac{\eta z^1 2}{\eta(83z)^1 2} - 83^6\frac{\eta(83z)^1 2}{\eta z^1 2}$
89	$\frac{\eta(z)^6}{\eta(89z)^6} - 89^3\frac{\eta(89z)^6}{\eta(z)^6}$
92	$\frac{\eta(4z)^4\eta(46z)^2}{\eta(2z)^2\eta(92z)^4} + 23\frac{\eta(2z)^2\eta(92z)^4}{\eta(4z)^4\eta(46z)^2}$
94	$\frac{\eta(2z)^8\eta(47z)^4}{\eta(z)^3\eta(94z)^8} - 47^2\frac{\eta(z)^3\eta(94z)^8}{\eta(2z)^8\eta(47z)^4}$
101	$\frac{\eta(z)^6}{\eta(101z)^6} - 101^3\frac{\eta(101z)^6}{\eta(z)^6}$
131	$\frac{\eta(z)^{12}}{\eta(131z)^{12}} - 131^6\frac{\eta(131z)^{12}}{\eta(z)^{12}}$

Table 4 (non-new case)

N	t
42	$\frac{\eta(z)^9\eta(2z)^3\eta(6z)^5\eta(14z)^3\eta(21z)^7}{\eta(3z)^{13}\eta(7z)^3\eta(42z)^{11}} - 7^2\frac{\eta(z)^3\eta(6z)^7\eta(7z)^3\eta(14z)^9\eta(21z)^5}{\eta(2z)^3\eta(3z)^{11}\eta(42z)^{13}}$
60	$\frac{\eta(2z)\eta(12z)^6\eta(20z)^2\eta(30z)^3}{\eta(4z)^2\eta(6z)^3\eta(10z)\eta(60z)^6} - 5\frac{\eta(2z)^3\eta(12z)^2\eta(20z)^6\eta(30z)}{\eta(4z)^6\eta(6z)\eta(10z)^3\eta(60z)^2}$
63	$\frac{\eta(9z)^3\eta(21z)}{\eta(3z)\eta(63z)^3} - 7\frac{\eta(3z)\eta(7z)^3}{\eta z)^3\eta(21z)}$
72	$\frac{\eta(z)^6\eta(6z)\eta(24z)^2\eta(36z)^3}{\eta(2z)^3\eta(3z)^2\eta(12z)\eta(72z)^6} - \frac{\eta(2z)^{15}\eta(3z)^2\eta(12z)\eta(24z)^2\eta(36z)^3}{\eta(z)^6\eta(4z)^6\eta(6z)^5\eta(72z)^6}$
75	$\frac{\eta(3z)^3\eta(25z)}{\eta(z)\eta(75z)^3} - 5^2\frac{\eta 3z)\eta(25z)^3}{\eta z)^3\eta(75z)}$
81	$\frac{\eta(z)^3\eta(27z)}{\eta(3z)\eta(81z)^3} - 3^5\frac{\eta(3z)\eta(81z)^3}{\eta(z)^3\eta(27z)}$
95	$\frac{\eta(5z)^5\eta(19z)}{\eta(z)\eta(95z)^5} - 19^2\frac{\eta(5z)\eta(19z)^5}{\eta(z)^5\eta(95z)}$
119	$\frac{\eta(7z)^7\eta(17z)}{\eta z)\eta(119z)^7} - 17^3\frac{\eta(7z)\eta(17z)^7}{\eta z)^7\eta(119z)}$

4.3. **Tables for $P(x,y)$.**

Table 5 (new case)

N	$P(x,y)$
34	$-48-32x+20x^2+24x^3+x^4+8(2+2x+x^2)y$
38	$-960+3168x+13160x^2+21724x^3+25833x^4+21810x^5+10071x^6+2065x^7+136x^8+x^9$ $+x(56+27x+x^2)(44+88x+137x^2+102x^3+17x^4)y$
43	$-7200-1680x+9400x^2-2332x^3-4868x^4+1708x^5+194x^6+x^7$ $-(-72+36x+x^2)(-44-84x+45x^2+22x^3)y$
44	$(2+x)^2(7+3x+5x^2+x^3)$
45	$x^2(-3+6x+x^2+4y)$
51	$-8904+89496x+720815x^2+2136731x^3+3806784x^4+4786996x^5+4564407x^6+3440158x^7$ $+2089704x^8+1029855x^9+409276x^{10}+129052x^{11}+31311x^{12}+5557x^{13}+658x^{14}+43x^{15}+x^{16}$ $+(-5+3x+2x^2+x^3)(36+37x+20x^2+3x^3)$ $(304+1445x+2641x^2+2567x^3+1636x^4+706x^5+201x^6+34x^7+2x^8)y$
53	$-247408+665520x-1831348x^2+4346036x^3-7515167x^4+7342874x^5-4503204x^6$ $+2095505x^7-818846x^8+230692x^9-33955x^{10}+739x^{11}+237x^{12}+x^{13}-(148-1108x$ $+151x^2+1363x^3-621x^4-12x^5+25x^6)(1328-1308x+805x^2-328x^3+47x^4+x^5)y$
54	$3-3x+3x^2+x^3+3(1+x)y$
55	$(2+x)(-138-271x-58x^2+1411x^3+168x^4-1461x^5-281x^6+349x^7+68x^8+x^9)$ $+(2+x)(-1+2x)(4+3x)(8-26x-78x^2+11x^3+28x^4+2x^5)y$
56	$(7+x^2)(2-x+x^2)(2+x+x^2)$
61	$-122+176x-27x^2-65x^3+18x^4+x^5-(1+x)(-22-15x+9x^2)y$
62	$-3840-448x+12724x^2+42628x^3+62861x^4+5174x^5+109639x^6+289900x^7+73179x^8$ $-61722x^9+143262x^{10}+178641x^{11}+61858x^{12}+7490x^{13}+253x^{14}+x^{15}+xy(-4+192x$ $+446x^2-108x^3-268x^4+443x^5+284x^6+23x^7)(32-56x+178x^2+248x^3+55x^4+x^5)$
64	$(-2+x)(2+x)(4+x^2)$
65	$1-35x-85x^2-15x^3-35x^4-50x^5-403x^6+10x^7+35x^8+65x^9-90x^{10}+25x^{11}+x^{12}$ $-5(-1+x)(1+x)(1+x^2)(-1-2x+x^2)(2-5x+2x^2+5x^3+2x^4)y$
69	$40128-2804032x-24658412x^2-82258148x^3-78001407x^4+286063638x^5+1082537261x^6$ $+1420597832x^7+9621058x^8-2694511846x^9-4047900698x^{10}-2330523372x^{11}+840632694x^{12}$ $+2638911745x^{13}+2331856822x^{14}+1199425309x^{15}+393442428x^{16}+82393205x^{17}+10602593x^{18}$ $+779531x^{19}+28810x^{20}+412x^{21}+x^{22}+y(-432-2920x-8036x^2-4860x^3+11574x^4+21734x^5$ $+14665x^6+4288x^7+477x^8+14x^9)(1276+3128x-4870x^2-28854x^3-3924x^4-657x^5+3703x^6$ $+40883x^7+1804x^8+3299x^9+202x^{10}+2x^{11})$
79	$-24843-7420x+112556x^2+76149x^3-214447x^4-113728x^5+157812x^6+73431x^7$ $-49467x^8-22769x^9+5008x^{10}+2736x^{11}+181x^{12}+x^{13}$ $-(1+x)(83-145x-10x^2+39x^3+x^4)(105+599x-75x^2-604x^3-31x^4+141x^5+21x^6)y$
83	$-846820980000+1701842643824x-4190038951864x^2-15407944317740x^3+52631374705524x^4$ $+195902048285636x^5-69755046878975x^6-1014877154551415x^7-1063602170418749x^8$ $+1855157981145929x^9+5075380899888979x^{10}+1636529117010692x^{11}-8302874421713802x^{12}$ $-11678206852543817x^{13}+1005402492172935x^{14}+17071231491541350x^{15}+14609656638884595x^{16}$ $-5861471722333698x^{17}-19444452135637043x^{18}-10558933244522770x^{19}+7380101893789387x^{20}$ $+13072119010688686x^{21}+4488232204563914x^{22}-4533961869101651x^{23}-5251219501592566x^{24}$ $-1152837411146407x^{25}+1460566880152011x^{26}+1235263008419117x^{27}+207233373480590x^{28}$ $-227667918937852x^{29}-159972030244333x^{30}-31212179742782x^{31}+11503837595608x^{32}$ $+9040609302177x^{33}+2671633081498x^{34}+434275870731x^{35}+39994913022x^{36}+2002817221x^{37}$ $+50297678x^{38}+551168x^{39}+1991x^{40}+x^{41}-y(452104-2937060x-9114834x^2+6837837x^3$ $+31676870x^4+15747540x^5-39815725x^6-52447587x^7+3221373x^8+46993123x^9+26434445x^{10}$ $-10569436x^{11}-16815463x^{12}-4439206x^{13}+2240588x^{14}+1704048x^{15}+372443x^{16}+26456x^{17}$ $+483x^{18}+x^{19})(1809956-279348x-16404210x^2-14857887x^3+44872703x^4+91386412x^5$ $+7209319x^6-135817396x^7-134605986x^8+24499468x^9+132852372x^{10}+79321327x^{11}$ $-23111323x^{12}-50396693x^{13}-19474566x^{14}+4349474x^{15}+6065443x^{16}+1970373x^{17}+236111x^{18}$ $+8734x^{19}+65x^{20})$

89	$-17600+410400x-27548480x^2+10400948x^3+146498188x^4-32027037x^5-360910680x^6$ $-17199072x^7+501894798x^8+161240831x^9-391130731x^{10}-233566274x^{11}+145795788x^{12}$ $+151162884x^{13}-4448240x^{14}-44442246x^{15}-12762946x^{16}+3535663x^{17}+2568284x^{18}$ $+483388x^{19}+30642x^{20}+515x^{21}+x^{22}-y(-1+x)(1+x)(2700+3240x-3609x^2-4873x^3$ $+323x^4+1473x^5+465x^6+17x^7)(648-1340x-10451x^2+10324x^3+23063x^4-2172x^5$ $-17100x^6-5730x^7+3597x^8+2256x^9+243x^{10}+2x^{11})$
92	$(3+x)^2(8+4x+4x^2+x^3)(8+20x+28x^2+25x^3+14x^4+5x^5+x^6)$
94	$-21715-47508x-103195x^2+219398x^3+1663909x^4+5469799x^5+10685097x^6$ $+13118353x^7+4598983x^8-18554364x^9-49262084x^{10}-67097732x^{11}-54688267x^{12}$ $-12748826x^{13}+34980862x^{14}+61712870x^{15}+58008344x^{16}+36328896x^{17}+15246919x^{18}$ $+3869878x^{19}+504954x^{20}+27667x^{21}+465x^{22}+x^{23}+y(-120-313x-845x^2-566x^3$ $+598x^4+2600x^5+3476x^6+2830x^7+1066x^8+105x^9+x^{10})(42-275x-1064x^2-2649x^3$ $-3694x^4-2814x^5+602x^6+4402x^7+5384x^8+3641x^9+814x^{10}+31x^{11})$
101	$3973376+24345712x-57185964x^2-178101464x^3+161167773x^4+462091312x^5$ $-328497531x^6-579757947x^7+504756351x^8+320809112x^9-453760989x^{10}+7488095x^{11}$ $+195974239x^{12}-88118541x^{13}-19312382x^{14}+31143884x^{15}-10685229x^{16}-391854x^{17}$ $+1833695x^{18}-900817x^{19}+273859x^{20}-59884x^{21}+9221x^{22}-857x^{23}+29x^{24}+x^{25}$ $-y(-2+x)(-2492-5984x+5085x^2+12629x^3-5462x^4-7457x^5+4351x^6+561x^7$ $-1100x^8+476x^9-115x^{10}+12x^{11})(-1164+4596x+5916x^2-11845x^3+1691x^4+5276x^5$ $-3587x^6+849x^7-17x^8-12x^9-4x^{10}+x^{11})$
131	$-11491793287200-77827916513200x+271274799348200x^2+2891920043667900x^3$ $-7413871118879700x^4-20818132963426070 0x^5+1446446595356863725x^6-676025577299157695x^7$ $-3427811900322703776 5x^8+2044381011226576604 50x^9-5685176548615368911 20x^{10}$ $+48154977972916451694 1x^{11}+26352457663686752058 51x^{12}-13175502704057926427 955x^{13}$ $+31767041079352267060729x^{14}-43173166155030136041073x^{15}+89690303714971621289 39x^{16}$ $+11242975823462958448 1836x^{17}-31037065461580741101 6501x^{18}+47093764141899259788 5012x^{19}$ $-40583968812222030704 5126x^{20}-80337487953419709312 71x^{21}+66861955557657262919 4039x^{22}$ $-12341095129507597346 21090x^{23}+13273194626488143699 90544x^{24}$ $-82256630805588320967 7805x^{25}-33736964013542515967 889x^{26}+78802788498091915526 8540x^{27}$ $-10931717302356296941 96110x^{28}+91100488154651137005 9093x^{29}-46755455518609590977 7550x^{30}$ $+47296265160342002570 173x^{31}+18525014304843260936 3127x^{32}-22596050904314156184 5001x^{33}$ $+16085263388637380938 5204x^{34}-77496232791333991087 714x^{35}+20802958115704540858 490x^{36}$ $+38800113884021811681 18x^{37}-87351925949078030990 43x^{38}+62107550075127122518 97x^{39}$ $-30034131633193485312 25x^{40}+10700206576670427672 23x^{41}-25940344938854044668 5x^{42}$ $+20958569399527863492x^{44}-14326879082941335638x^{45}+5912238306538064916x^{46}$ $+1847795392183094798 5x^{43}-1877971630289666366x^{47}+488283858739094050x^{48}$ $-105722203629934910x^{49}+18923329741114761x^{50}-2689272320340289x^{51}+263300065498249x^{52}$ $-42605056366 61x^{53}-5157557959005x^{54}+1416554294641x^{55}-243661620624x^{56}+31240505016x^{57}$ $-2997027299x^{58}+189957325x^{59}-2699879x^{60}-1016956x^{61}+130698x^{62}-7747x^{63}+187x^{64}+x^{65}$ $-y(-1+x)(-1651336-45600164x-27727502x^2+1818599597x^3-7927670906x^4$ $+13960778588x^5-421393207x^6-51398069731x^7+119271135931x^8-134713281163x^9$ $+52661398574x^{10}+80481104397x^{11}-163994123959x^{12}+148050275877x^{13}-72614888314x^{14}$ $+6183551272x^{15}+20717235269x^{16}-19147906676x^{17}+9847397087x^{18}-3367121265x^{19}$ $+730146720x^{20}-51809318x^{21}-30959822x^{22}+15666228x^{23}-4292594x^{24}+835815x^{25}-122075x^{26}$ $+13084x^{27}-903x^{28}+23x^{29}+x^{30})(-4255900+71222100x+1461150x^2-2150436025x^3$ $+7702741575x^4-2543947440x^5-421393207x^6-51398069731x^7+119271135931x^8$ $-134713281163x^9+52661398574x^{10}+80481104397x^{11}-163994123959x^{12}+148050275877x^{13}$ $-72614888314x^{14}+6183551272x^{15}+20717235269x^{16}-19147906676x^{17}+9847397087x^{18}$ $-3367121265x^{19}+730146720x^{20}-51809318x^{21}-30959822x^{22}+15666228x^{23}-4292594x^{24}$ $+835815x^{25}-122075x^{26}+13084x^{27}-903x^{28}+23x^{29}+x^{30})(-4255900+71222100x+1461150x^2$ $-2150436025x^3+7702741575x^4-2543947440x^5-55631723695x^6+194169167040x^7$ $-323648869980x^8+226360273540x^9+229278613251x^{10}-835109705023x^{11}+1135514398082x^{12}$ $-867152409818x^{13}+237463102523x^{14}+300544359512x^{15}-479908400891x^{16}+368024466511x^{17}$ $-177471360585x^{18}+47572071000x^{19}+3135089681x^{20}-10639183810x^{21}+6239233500x^{22}$ $-2354646874x^{23}+659880730x^{24}-144046879x^{25}+25143814x^{26}-3620625x^{27}+455809x^{28}$ $-54178x^{29}+6052x^{30}-527x^{31}+24x^{32})$

Table 6 (non-new case)

N	$P(x,y)$
42	$9(4+x+4x^2)\big(64+1017x+96x^2+1178x^3-1352x^4+2883x^5-1336x^6+730x^7+1800x^8$ $+1417x^9+64x^{10}+72y(-1+x)(1+x)(50+213x-6x^2+215x^3-6x^4+213x^5+50x^6)\big)$
60	$(-1+x+x^2)(-1+4x+x^2)(1-x+2x^2+x^3+x^4)$
63	$1+9x-24x^2-8x^3-9x^4+3x^5+x^6-3(-2+x)(-1+x)(1+x)(-1+2x)y$
72	$4(7+144x+72x^2+72x^3+144x^4+72x^5)+144(1+x)(-2+6x+x^3)y$
75	$-99-390x-569x^2-372x^3-67x^4+56x^5+40x^6+11x^7$ $-3x(2+x)(-2-4x+3x^2+5x^3+2x^4)y$
81	$-968+132x+837x^2-3213x^3+4107x^4-2223x^5+510x^6-21x^7-9x^8+x^9$ $-3(-1+x)(-29+27x-9x^2+x^3)(11+24x-15x^2+2x^3)y$
95	$x(1-1208x-10934x^2-44162x^3-109477x^4-180353x^5-196536x^6-134741x^7-40197x^8+$ $17286x^9+20552x^{10}+4156x^{11}-2604x^{12}-1110x^{13}+370x^{14}+18x^{15}-13x^{16}+x^{17})$ $+(1+x)(-1-7x-1218x^2-8470x^3-25928x^4-48038x^5-51018x^6-26540x^7+$ $2089x^8+10997x^9+2802x^{10}-1822x^{11}-545x^{12}+323x^{13}-46x^{14}+2x^{15})y$
119	$-19456+128127x+31684x^2-1935597x^3+2686286x^4+12402399x^5-29855351x^6-41030815x^7$ $+159290916x^8+60745925x^9-524982545x^{10}+38052811x^{11}+1173318320x^{12}-347340496x^{13}$ $-1864043953x^{14}+721098268x^{15}+2154666360x^{16}-836975880x^{17}-1819472378x^{18}$ $+598982372x^{19}+1104865348x^{20}-257570193x^{21}-464230322x^{22}+55914351x^{23}+126047469x^{24}$ $-504853x^{25}-19453402x^{26}-2191709x^{27}+1202359x^{28}+297037x^{29}+20766x^{30}+407x^{31}$ $-(1+x)(19457-147570x+135420x^2+1672260x^3-4351085x^4-6352626x^5+33446252x^6$ $-1612436x^7-131903527x^8+93891838x^9+312797860x^{10}-358809060x^{11}-481203790x^{12}$ $+724102984x^{13}+508610138x^{14}-917130440x^{15}-392684774x^{16}+764808468x^{17}+236037174x^{18}$ $-417563192x^{19}-113096097x^{20}+141052022x^{21}+40143942x^{22}-25553396x^{23}$ $-8853365x^{24}+1504686x^{25}+853274x^{26}+100016x^{27}+3681x^{28}+30x^{29})y$

References

[1] D. Abramovich and J. Harris. Abelian varieties and curves in $W_d(C)$. *Compositio Math.*, 78(2):227–238, 1991.

[2] A. O. L. Atkin and J. Lehner. Hecke operators on $\Gamma_0(m)$. *Math. Ann.*, 185:134–160, 1970.

[3] F. Bars. Bielliptic modular curves. *J. Number Theory*, 76(1):154–165, 1999.

[4] P. Bayer and A. Travesa, editors. *Corbes modulars: Taules.* Notes del Seminari de Teoria de Nombres de Barcelona, UB-UAB-UPC, Barcelona, 1992.

[5] B. J. Birch and W. Kuyk, editors. *Modular functions of one variable. IV.* Lecture Notes in Mathematics, Vol. 476. Springer-Verlag, Berlin, 1975.

[6] J. Gonzàlez Rovira. Equations of hyperelliptic modular curves. *Ann. Inst. Fourier (Grenoble)*, 41(4):779–795, 1991.

[7] M. A. Kenku and F. Momose. Automorphism groups of the modular curves $X_0(N)$. *Compositio Math.*, 65(1):51–80, 1988.

[8] J. Lehner and M. Newman. Weierstrass points of $\Gamma_0\,(n)$. *Ann. of Math. (2)*, 79:360–368, 1964.

[9] A. P. Ogg. Hyperelliptic modular curves. *Bull. Soc. Math. France*, 102:449–462, 1974.

Josep González,
Universitat Politècnica de Catalunya,
Departament de Matemàtica Aplicada IV (EPSEVG),
Av. Victor Balaguer s/n,
08800 Vilanova i la Geltrú, Spain.
E-mail address: `josepg@ma4.upc.edu`